separation is achieved by paying the maximum penalty in weight, a conclusion which differs from Refs. 1 and 2.

The characteristic of the design with the maximum lowest frequency is rather similar to that with the largest frequency separation. The former has a somewhat higher lowest frequency but no significant separation of this frequency from the second or third lowest frequencies. Here again, shell panel vibration controls the design.

Comparing the designs in Table 1, it appears that rather similar designs may behave quite differently with respect to frequency separation. This raises a serious question with regard to the validity of employing the frequency prediction models used here for design for optimal frequency separation of immersed shells. Orthotropic shell theory does not provide an extremely accurate approximation to actual behavior. Furthermore, this and the earlier studies of Refs. 1 and 2 use in vacuo frequencies to study the characteristics of submerged shells. Considering the inaccuracy in these models and the sensitivity of the frequency separation results to relatively small design changes, the results do not appear to be useful for design purposes.

In addition to the search starting point cited in Table 1, runs were made using the six variable formulations for problem types A and B from the three additional starting points. These also converged to designs similar to those in Table 1; thus the starting point sensitivity noted in Ref. 1 on their problem type B was not apparent in this study. These results indicate that the optimization procedure used here is apparently capable of locating an optimum design with reasonable reliability for the problem types studied.

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A Minimum Mass Square Plate with Fixed Fundamental Frequency of Free Vibration

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Introduction

THE minimum mass design of structural members subject to dynamic behavioral constraints has been the focus of considerable research effort recently.[1] Constraints on one or more natural frequencies of free vibration or flutter speed are often imposed in the design of major structural components for lightweight, high-performance aircraft and aerospace vehicles.

One approach to the minimum mass design of basic structural members, such as a beam, a column, or a plate, is to represent the member by a continuous model whose behavior is described by a differential equation. The optimal design is determined by applying methods based on the calculus of variations or its extension in the form of optimal control theory.

The application of methods from the theory of optimal control has proven to be a very powerful technique when the nature of the structural member is such that its behavior can be described by an ordinary differential equation in one independent spatial variable (one-dimensional structure). Relatively few applications have been made to the optimal design of structural members whose behavior is described by a partial differential equation in two independent spatial variables (two-dimensional structures). In control theory terminology, these would be classified as distributed parameter optimal control problems.

The purpose of this Note is to illustrate the application of a simple computational technique to the problem of determining the minimum mass design of a simply supported square plate with fixed fundamental frequency of free vibration. Previous efforts to solve this problem[2-4] have utilized methods requiring the numerical solution of a boundary value problem characterized by the partial differential equation modeling the behavior of the plate. This is an extremely complex and time-consuming process. One theoretically promising technique for solving optimal control problems that avoids explicitly solving the governing differential equation for the system is the ϵ method of Balakrishnan.[5] The basic idea of this method is to replace the differential equation modeling the system by a penalty function, thus transforming the original dynamic problem into a nondynamic one. Foley and Citron[6] have applied a technique based on the ϵ method to one-dimensional structural optimization problems. This Note presents an extension of that technique to a two-dimensional structural optimization problem.

Formulation of the Optimal Design Problem

The design objective is to continuously vary the thickness, $T(X,Y)$, of the plate over the region $\{(X,Y) \mid 0 \le X \le a, \quad 0 \le Y \le a\}$ so as to minimize the mass of the plate, while keeping the fundamental frequency of free vibration fixed and equal to that of a reference plate of uniform thickness, T_0, and identical dimensions. The behavior of the plate may be

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described by the following system of four second-order partial differential equations.[7]

$$\frac{\partial^2 W}{\partial X^2}+\frac{12}{ET^3}(M_X-\nu M_Y)=0 \tag{1a}$$

$$\frac{\partial^2 W}{\partial Y^2}+\frac{12}{ET^3}(M_Y-\nu M_X)=0 \tag{1b}$$

$$0\le X\le a,\quad 0\le Y\le a$$

$$\frac{\partial^2 W}{\partial X\partial Y}-12\frac{(1+\nu)}{ET^3}M_{XY}=0 \tag{1c}$$

$$\frac{\partial^2 M_X}{\partial X^2}+\frac{\partial^2 M_Y}{\partial Y^2}-2\frac{\partial^2 M_{XY}}{\partial X\partial Y}+\rho T\omega_0^2 W=0 \tag{1d}$$

where M_X and M_Y are the bending moments, M_{XY} is the twisting moment, W is the deflection, ρ is the mass density, ν is Poisson's ratio, and ω_0 is the fixed fundamental frequency of free vibration. Introducing the nondimensional coordinates $x=X/a$, $y=Y/a$, the nondimensional state variables $z_1=M_Xa/ET_0^3$, $z_2=M_Ya/ET_0^3$, $z_3=M_{XY}a/ET_0^3$, and $z_4=W/a$, and the nondimensional control variable $u=T/T_0$, the system Eq. (1) can be written in operator form as:

$$L[u]z=\begin{bmatrix}\dfrac{\partial^2 z_4}{\partial x^2}+\dfrac{12}{u^3}(z_1-\nu z_2)\\ \dfrac{\partial^2 z_4}{\partial y^2}+\dfrac{12}{u^3}(z_2-\nu z_1)\\ \dfrac{\partial^2 z_4}{\partial x\partial y}-12\dfrac{(1+\nu)}{u^3}z_3\\ \dfrac{\partial^2 z_1}{\partial x^2}+\dfrac{\partial^2 z_2}{\partial y^2}-2\dfrac{\partial^2 z_3}{\partial x\partial y}\end{bmatrix}$$

$$=-\frac{\pi^4}{3(1-\nu^2)}\begin{bmatrix}0\\0\\0\\uz_4\end{bmatrix}=\zeta Q[u]z \tag{2}$$

where $(x,y)\in\Omega=\{(x,y)\,|0\le x\le 1, 0\le y\le 1\}$.

The boundary conditions for simply supported edges are

$$z_1=0 \text{ for } x=0 \text{ and } x=1 \tag{3a}$$

$$z_2=0 \text{ for } y=0 \text{ and } y=1 \tag{3b}$$

$$z_4=0 \text{ for } x=0,\ y=0,\ x=1,\ y=1 \tag{3c}$$

The optimal design problem can be formulated as the following distributed parameter optimal control problem.

Determine the control variable $u(x,y)$ which minimizes the mass criterion

$$J=\iint_\Omega u(x,y)\,\mathrm{d}x\mathrm{d}y$$

subject to the state Eq. (2), the boundary conditions, Eqs. (3), and

$$g[u]=u_{\min}-u(x,y)\le 0 \tag{4}$$

This problem will be referred to as problem A.

The control variable inequality constraint, Eq. (4), is imposed to prevent the nonrealistic situation of zero thickness from occurring. It is interesting to observe the natural definition of the state variables in terms of the physical response of the plate.

Mathematical Description of the Technique

The ϵ problem is formulated as follows: For any $\epsilon_1>0$ and $\epsilon_2>0$, determine the continuous function $u(x,y)$ which minimizes

$$J_{\epsilon_1,\epsilon_2}=\iint_\Omega\Big[u(x,y)+\frac{1}{\epsilon_1}\|L[u]z-\zeta Q[u]z\|^2+\frac{1}{\epsilon_2}\max{}^2\{0,g[u]\}\Big]\mathrm{d}x\mathrm{d}y$$

where

$$z_1(0,y)=z_1(1,y)=z_2(x,0)=z_2(x,1)=0$$

$$z_4(0,y)=z_4(1,y)=z_4(x,0)=z_4(x,1)=0$$

This problem will be referred to as problem B.

One would expect that, under the appropriate conditions, as $\epsilon_1\to 0$ and $\epsilon_2\to 0$ sequentially, the corresponding sequence of optimal solutions of problem B will converge to the optimal solution of problem A. This hypothesis is supported by the work of Balakrishnan,[5] who has developed the theory for optimal control problems with ordinary differential equation constraints, and Lions,[8] who has applied such an approach to optimal control problems with partial differential equation constraints.

Let $\{\phi_{mn,k}(x,y)\}$ denote a set of functions which is complete in the region Ω, and which satisfies the appropriate boundary conditions for the state variable z_k. Expand each of the state variables in a series of the form

$$z_k^{M,N}(x,y)=\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_{mn,k}\phi_{mn,k}(x,y)$$

where the $\alpha_{mn,k}$ are undetermined coefficients. In this case, each of the state variables is expanded as follows:

$$z_1^{M,N}=\sum_{m=1}^{M}\sum_{n=1}^{N}a_{mn}\sin(2m-1)\pi x\,\sin(2n-1)\pi y \tag{5a}$$

$$z_2^{M,N}=\sum_{m=1}^{M}\sum_{n=1}^{N}b_{mn}\sin(2m-1)\pi x\,\sin(2n-1)\pi y \tag{5b}$$

$$z_3^{M,N}=\sum_{m=1}^{M}\sum_{n=1}^{N}c_{mn}\cos(2m-1)\pi x\,\cos(2n-1)\pi y \tag{5c}$$

$$z_4^{M,N}=\sum_{m=1}^{M}\sum_{n=1}^{N}d_{mn}\sin(2m-1)\pi x\,\sin(2n-1)\pi y \tag{5d}$$

The coefficient d_{11} is chosen as unity, since the deflection of the reference plate is only determined up to a multiplicative constant. Due to symmetry, only modes which are odd in both coordinates are retained, and $a_{mn}=a_{nm}$, $b_{mn}=b_{nm}$, $c_{mn}=c_{nm}$, and $d_{mn}=d_{nm}$.

Consider the space of state functions (denoted S_{mn}) whose components are of the form of Eq. (5). Any admissible state vector z for problem B can be approximated as closely as desired by functions in S_{mn} for sufficiently large M and N. The ϵ problem over the space S_{mn} is formulated as follows: For $\epsilon_1>0$ and $\epsilon_2>0$, determine the continuous function $u(x,y)$ and the vector of coefficients $\boldsymbol{\alpha}_{mn}$ which minimizes

$$J_{\epsilon_1,\epsilon_2}^{M,N}=\iint_\Omega\Big[u(x,y)+\frac{1}{\epsilon_1}\|L[u]z^{M,N}-\zeta Q[u]z^{M,N}\|^2+\frac{1}{\epsilon_2}\max{}^2\{0,g[u]\}\Big]\mathrm{d}x\mathrm{d}y \tag{6}$$

This problem will be referred to as problem C.

Intuitively, one would expect that as $\epsilon_1, \epsilon_2 \to 0$ and $M,N \to \infty$ sequentially, the corresponding sequence of optimal solutions of problem C will converge to the optimal solution of problem A. Balakrishnan[9] discussed the computational aspects of this approach for the one-dimensional analog of problem A.

Numerical Results

In order to obtain computational results, the problem is reformulated as a parameter optimization problem. The region Ω is subdivided into a square grid with spacing h. From symmetry considerations, a solution may be obtained for only one quadrant of the plate, thus reducing the dimensionality of the problem. The lines of the grid for the selected quadrant are $x_i = (i-1)h$, $i=1,\ldots,K$, and $y_j = (j-1)h$, $j=1,\ldots,K$, where $h = 1/2(K-1)$. Discretizing and utilizing the trapezoidal rule for numerical integration, the performance criterion Eq. (6), can be approximated by the objective function

$$J_{\epsilon_1,\epsilon_2}^{M,N} \cong h^2[f_{1,1} + f_{K,1} + f_{1,K} + f_{K,K}] + 2h^2\left[\sum_{i=2}^{K-1}(f_{i,1} + f_{i,K}) + \sum_{j=2}^{K-1}(f_{1,j} + f_{K,j})\right] + 4h^2\sum_{i=2}^{K-1}\sum_{j=2}^{K-1} f_{i,j} \tag{7}$$

where

$$f_{i,j} = u(x_i,y_j) + (1/\epsilon_1)\|L[u]z^{M,N} - \zeta Q[u]z^{M,N}\|^2_{x_i,y_j} + (1/\epsilon_2)\max^2\{0, g[u(x_i,y_j)]\}$$

A decision vector is defined by

$$c = \begin{bmatrix} \alpha_{mn} \\ \hline u_{ij} \end{bmatrix}$$

where α_{mn} is the vector of undetermined coefficients in the state variable expansions, Eqs. (5), and u_{ij} is the vector whose components are the values of the nondimensional thickness distribution at the nodes of the grid.

It can be shown that the objective function, Eq. (7), may be expressed as a sum of the squares of a set of functions $F_k(c)$. Thus, the original distributed parameter optimal control problem (problem A) has now been transformed into a least-squares parameter optimization problem. The objective is to determine, for any $\{\epsilon_1, \epsilon_2, M, N, K\}$, the vector c which minimizes

$$\sum_k F_k^2(c)$$

This allows the method of Gauss-Levenberg-Marquardt to be employed, which is one of the most powerful available for least-squares problems.

Numerical solutions were obtained using a commercially available Gauss-Levenberg-Marquardt algorithm (IMSL Library subroutine *ZXSSQ*) for a 49-node grid. Following the spirit of the method of collocation, the number of terms in each state-variable expansion was chosen to correspond to the number of nodes (x_i,y_j), $i=2,\ldots,K$, $j=2,\ldots,K$, where the

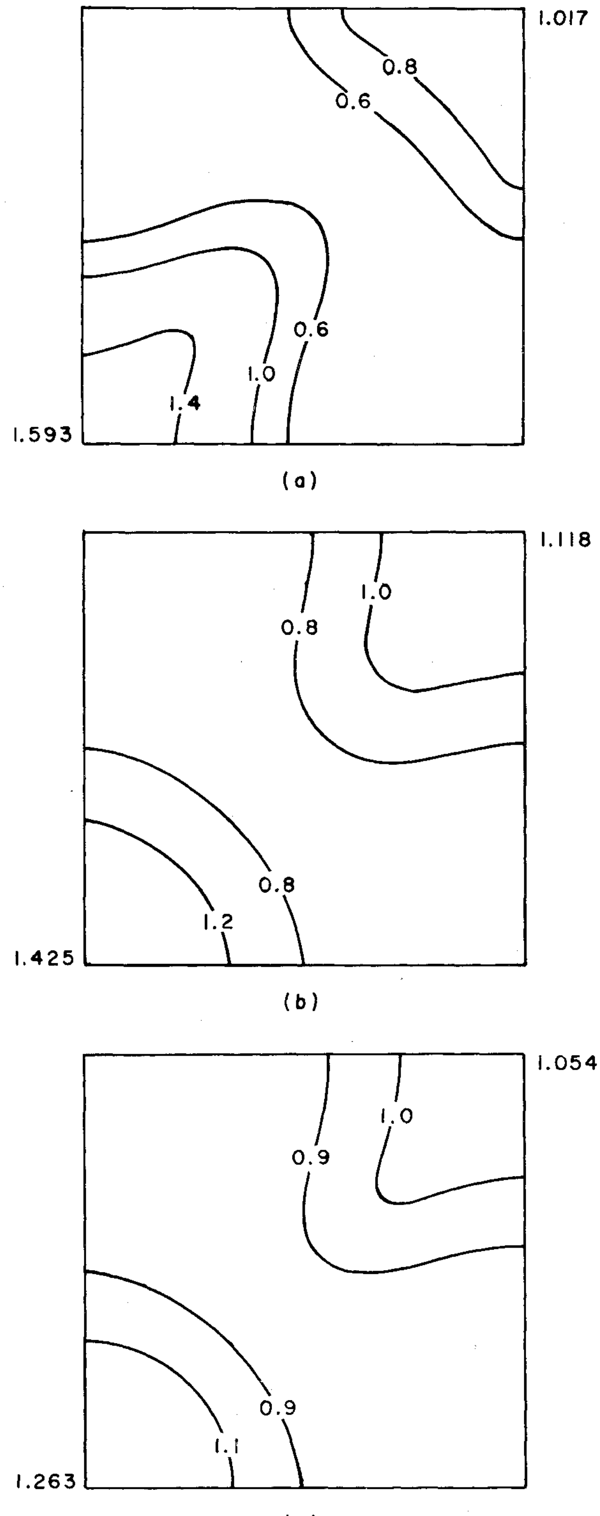

Fig. 1 Contour lines of the optimal nondimensional thickness distribution for one quadrant of a simply supported square plate with fixed fundamental frequency of free vibration. a) $u_{min} = 0.6$, b) $u_{min} = 0.8$, c) $u_{min} = 0.9$.

Table 1 Minimum mass simply supported square plate with fixed fundamental frequency of free vibration

u_{min}	M	N	K	ϵ_1	ϵ_2	Mass ratio	Objective function	Iterations for convergence
0.6	3	3	4	1.0	0.001	0.80574	0.80616	83
0.8	3	3	4	1.0	0.001	0.91218	0.91252	72
0.9	3	3	4	1.0	0.001	0.95754	0.95789	79

Table 2 Optimal nondimensional thickness distribution $u(x,y)$ for one quadrant of a simply supported square plate with fixed fundamental frequency of free vibration

x/y	0.0	0.167	0.333	0.5
		$u_{min}=0.6$		
0.0	1.593	1.161	0.6	0.6
0.167	1.161	1.253	0.6	0.6
0.333	0.6	0.6	0.6	0.888
0.5	0.6	0.6	0.888	1.017
		$u_{min}=0.8$		
0.0	1.425	1.196	0.8	0.8
0.167	1.196	0.8	0.8	0.8
0.333	0.8	0.8	0.995	0.984
0.5	0.8	0.8	0.984	1.118
		$u_{min}=0.9$		
0.0	1.263	1.103	0.9	0.9
0.167	1.103	0.9	0.9	0.9
0.333	0.9	0.9	1.005	0.982
0.5	0.9	0.9	0.982	1.054

Table 3 Effect of variation of the parameters ϵ_1 and ϵ_2 on the convergence of the method

ϵ_1	ϵ_2	Mass ratio	Objective function	Iterations
100.0	0.0001	0.91364	0.93501	111
10.0	0.0001	0.95315	0.95567	88
1.0	0.0001	0.95770	0.95796	100
0.1	0.0001	0.95816	0.95820	109
0.01	0.0001	0.95821	0.95822	137
0.001	0.0001	0.95821	0.95822	166
1.0	0.1	0.94029	0.94935	54
1.0	0.01	0.95602	0.95713	69
1.0	0.001	0.95754	0.95789	79
1.0	0.0001	0.95770	0.95796	100

square error in satisfying the partial differential equation of constraint is to be minimized (the partial differential equation is automatically satisfied at nodes (x_i,y_j) where $i=1$ or $j=1$).

The results of varying the minimum thickness constraint are presented in Table 1. The mass ratio provides a means of determining the weight savings for the optimal design. It is defined as the ratio of the mass of the optimal plate to that of the reference plate. Thus, weight savings of over 4%, 8%, and 19% were realized for the three respective minimum thickness constraints. In each case, the iterations required for convergence are tabulated for an initial guess corresponding to the reference plate. Convergence was assumed when the magnitude of the gradient of the objective function was less than 0.0001. The values of the optimal thickness distribution, $u(x,y)$, at the nodes of the grid for one quadrant of the plate are presented in Table 2 for each of the three minimum thickness constraints. The general tendency was for the mass to build up significantly at the corners of the plate and slightly at the center. Contour lines of the optimal thickness distributions for one quadrant of the plate are plotted in Fig. 1.

Table 3 illustrates the results of varying the magnitude of the parameters ϵ_1 and ϵ_2 for a minimum thickness constraint of $u_{min}=0.9$. A larger value of ϵ_1 implies a less severe penalty for violation of the differential equation of constraint, thus allowing a less accurate optimal thickness distribution and a smaller and less accurate value of the mass ratio. The value of the parameter ϵ_2 had to be kept relatively small in order to accurately satisfy the minimum thickness constraint. For practical purposes, a sufficiently accurate optimal solution can be obtained for $\epsilon_1=1.0$ and $\epsilon_2=0.001$.

Conclusions

The technique described herein has proven itself effective for the solution, via optimal control methodology, of an optimal structural design problem involving a two-dimensional structural member. The principle advantages are:

1) It avoids the direct solution of the partial differential equation of constraint.

2) The boundary conditions are automatically satisfied.

3) It is conceptually simple and easy to implement.

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Effect of Temperature-Dependent Heat Capacity on Aerodynamic Ablation of Melting Bodies

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Nomenclature

c	= heat capacity per unit volume of the material
c_0	= heat capacity at melting temperature
h	= heat-transfer coefficient
H	= heat flow vector
k	= thermal conductivity of the material
L	= latent heat of the material
$q_1(t)$	= unknown surface temperature
$q_2(t)$	= unknown melting distance
t	= time
T_f	= temperature of the surroundings
α	= thermal diffusivity
β	= dimensionless temperature of the surroundings, c_0T_f/ρ_mL
γ	= dimensionless coefficient to give variation in heat capacity, λT_f
η	= dimensionless melting distance, hq_2/k

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